Enterprise investment under price-sensitive demand

Xie Xiang-tian *

Key Lab of Science & Technology in Finance, Guangdong University of Finance, Guangzhou, China, 510521

122627748@qq.com

Keywords: Real options; Investment under uncertainty; Equity financing; Debt financing; Price-sensitive demand

Abstract. This paper proposed an equity financing investment model and a debt financing investment model based on real options theory, with considering price-sensitive demand and price obeying the Geometric Brownian Motion. We derived the optimal solutions, which show that the optimal bankruptcy threshold of debt financing is higher than equity financing, the optimal bankruptcy threshold and the optimal investment threshold decrease as the market capacity increases, and increase as the price-sensitive coefficient increases.

Introduction

With more uncertainty of market environment, a growing number of scholars study investment based on real options theory. Dixit and Pindyck (1994) proposed that uncertain investment has three characteristics: (1)irreversibility;(2)delayability;(3)uncertainty over the future returns. The returns on invested assets are not only related to price, but also to demand. In general, demand decreases as price rises ,and vice versa. So how does this inverse relationship affect investment? Considering price-sensitive demand, This paper built up two investment models. By solving the models, the optimal bankruptcy threshold expression and the optimal investment threshold equation were obtained.

Related literature. As far as I know, McDonald and Siegel (1986) considered the relationship of demand and price to study investment earlier,they constructed a nonlinear function of price about demand, and obtained that the option value increases as the demand uncertainty increases in the example of a monopolist investment decision. Subsequently, Kulatilaka and Perotti (1998) constructed a linear function of price about demand and demand random increasing variable, and proved that demand uncertainty would cause enterprises to postpone the execution of growth options.Recently, some scholars constructed a function of price about demand, which is like to Kulatilaka and Perotti's (1998) to study capacity investment,they demonstrated that enterprises with flexible capacity would implement capacity expansion earlier(Dangl (1999), Hagspiel V, etc. (2016)), De and Massabò (2018)).Different from the above literature, this paper constructed a function of demand about price to study uncertain investment.

Model

Considering that an enterprise has a growth option, the enterprise will pay for the investment cost through equity financing or debt financing, setting the investment cost as I. The enterprise executes the growth option according to the market price, and setting the market price as p,assuming the market price obeies the Geometric Brownian Motion, then

$$dp=\mu pdt+\sigma pdW \tag{1}$$

Where μ is called the drift parameter,and σ the variance parameter, dW is the increment of the Wiener process: $dW=\varepsilon(t)(dt)^{1/2}$, $\varepsilon(t)$ is the continuous random variable on t,$\varepsilon(t)$~N(0,1) and E($\varepsilon(i)$, $\varepsilon(j)$)=0 for any i, $j(i\neq j)$.

Because demand decreases as price rises, assuming demand about price is a linear reduction function: $D(p)=a-bp$, where a is the demand when b=0, which is called market capacity, b is the

  DOI: 10.25236/icess.2019.209

price-sensitive coefficient of the demand. Setting τ as the corporate tax rate and Z as the operating cost, then the after-tax net income is

$R(p)=(1-\tau)[p(a-bp)-Z]$ (2)

Setting other symbols: r=discount rate, in order to ensure the bankruptcy threshold is positive, assuming $r>2\mu+\sigma^2$; p_i= the optimal investment threshold under equity financing; p_s=the optimal bankruptcy threshold under equity financing; p_{ci}= the optimal investment threshold under debt financing; p_{cs}= the optimal bankruptcy threshold under debt financing; c= the debt payment for each period; T_i= the time when the enterprise executes the growth option; T_s= the time when the enterprise executes the bankruptcy.

Equity financing.

Considering that the investment cost comes from equity financing when the enterprise executes the growth option, then the expected value of the enterprise before bankruptcy is

$$V(p)=E_t\int_t^{T_s}(1-\tau)\left[p(a-bp)-Z\right]e^{-r(s-t)}ds \quad p\ge p_s \tag{3}$$

Equation (3) is a dynamic programming, which represents the total net income in $[t, T_s]$. Equation (3) can be equivalently converted to the ordinary differential equation (Pindyck (1990)):

$$rV(p)=(1-\tau)[p(a-bp)-Z]+\mu pV'(p)+p^2\sigma^2V''(p)/2 \quad p\ge p_s \tag{4}$$

The investment cost comes from equity financing, so bankruptcy is due to operating deficit.After bankruptcy liquidation, the enterprise value is zero(Brennan, Schwartz(1984)), that is: $V(p_s)=0$. Therefore, the optimal bankruptcy threshold satisfies the following conditions:

$$V(p_s)=0 \tag{5}$$

$$V'(p_s)=0 \tag{6}$$

Where, the equation (6) is the smooth-pasting condition.

Proposition 1. If financing is through equity, the enterprise value before bankruptcy is

$$V(p)=\mathrm{H}(p)-\mathrm{H}(p_s)(p/p_s)^{\beta} \quad p\ge p_s \tag{7}$$

Where,

$$\mathrm{H}(p)=(1-\tau)\left(\frac{ap}{r-\mu}-\frac{bp^2}{r-2\mu-\sigma^2}-\frac{Z}{r}\right) \tag{8}$$

$$\beta=\frac{1}{\sigma^2}\left[-\left(\mu-\frac{\sigma^2}{2}\right)-\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right]<0 \tag{9}$$

$$p_s=(-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}})/(2\mathrm{A}) \tag{10}$$

$\mathrm{A}=br(\beta-2)(r-\mu)$, $\mathrm{B}=ar(\beta-1)(\sigma^2+2\mu-r)$, $\mathrm{C}=Z\beta(r-\mu)(r-2\mu-\sigma^2)$.

Proof. Because the general solution of the equation (4) is: $V(p)=\mathrm{H}(p)-\mathrm{C}\_(p/p_s)^{\beta}$, according to $V(p_s)=0$, $\mathrm{C}\_=\mathrm{H}(p_s)$ can be obtained, and since $V'(p)=\mathrm{H}'(p)-\mathrm{H}(p_s)(\beta/p)(p/p_s)^{\beta}$, so $V'(p_s)=\mathrm{H}'(p_s)-\mathrm{H}(p_s)(\beta/p_s)$. Because $V'(p_s)=0$, $p_s=\{[-\mathrm{B}+(\mathrm{B}^2-4\mathrm{AC})^{1/2}]/(2\mathrm{A}),[-\mathrm{B}-(\mathrm{B}^2-4\mathrm{AC})^{1/2}]/(2\mathrm{A})\}$. At last, since A<0, so $p_s=[-\mathrm{B}+(\mathrm{B}^2-4\mathrm{AC})^{1/2}]/(2\mathrm{A})$.

Considering that the enterprise has a growth option, the equity value before executing the growth option ($0<t\le T_i, p\le p_i$) satisfies the following model:

$$rE_0(p)=\mu pE_0'(p)+p^2\sigma^2E_0''(p)/2 \quad p\le p_i$$
$$s.t.\begin{cases}E_0(p_i)=V(p_i)-\mathrm{I}\\E_0'(p_i)=V'(p_i)\end{cases} \tag{11}$$

Proposition 2. The equity value before executing the growth option is

$$E_0(p)=(V(p_i)-\mathrm{I})(p/p_i)^{\gamma} \quad p\le p_i \tag{12}$$

Where,

$$\gamma=\frac{1}{\sigma^2}\left[-\left(\mu-\frac{\sigma^2}{2}\right)+\sqrt{\left(\mu-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right]>1 \tag{13}$$

p_i satisfies the following equation:

$$(\mathrm{H}(p_i)-\mathrm{I})\gamma-(1-\tau)\left(\frac{ap_i}{r-\mu}-\frac{2bp_i^2}{r-2\mu-\sigma^2}\right)=(\gamma-\beta)\mathrm{H}(p_s)\left(\frac{p_i}{p_s}\right)^{\beta} \tag{14}$$

Proof. Because the general solution of the model(11) is $E_0(p)$=C_ $(p/p_i)^{\gamma}$, according to $E_0(p_i)=V(p_i)$- I,C_=V (p_i)-I can be obtained. Because $E_0{}'(p)$=[H(p_i)-H$(p_s)(p_i/p_s)^{\beta}$-I]$(\gamma/p)(p/p_i)^{\gamma}$=>$E_0{}'(p_i)$ =[H(p_i)-H(p_s) $(p_i/p_s)^{\beta}$-I](γ/p_i), and $V'(p)$=H$'(p)$-H$(p_s)(\beta/p)(p/p_s)^{\beta}$=>$V'(p_i)$ =H$'(p_i)$-H(p_s) $(\beta/p_i)(p_i/p_s)^{\beta}$,equation (14) can be obtained according to the smooth-pasting condition: $E_0{}'(p_i)=V'(p_i)$. Finally, Let F(p_i)=(H(p_i)-I) γ-$(1-\tau)[ap_i/(r-\mu)-2p_ib/(r-2\mu-\sigma^2)]$-$(\gamma-\beta)$ H$(p_s)(p_i/p_s)^{\beta}$, since $\lim_{p_i\to 0}$ F(p_i)=-∞ and $\lim_{p_i\to\infty}$ F(p_i)= ∞, equation(14)must have a real solution.

If the enterprise tax rate and price-sensitive demand are not considered, and the market capacity is assumed to be 1, i.e. τ=0,b=0 and a=1,then equation (14) is consistent with the equation satisfying the optimal investment threshold of Bolton, Wang and Yang (2014).

Debt financing.

Considering that the investment cost comes from debt financing when the enterprise executes the growth option, then the expected value of the equity before executing the bankruptcy is

$$E(p)=E_t\int_t^{T_s}(1-\tau)\left[p\left(p-bp\right)-Z-c\right]e^{-r(s-t)}ds \quad p\ge p_{cs} \tag{15}$$

Equation (15) can be equivalently converted to the ordinary differential equation:

$$rE(p)=(1-\tau)[p(a-bp)-Z-c]+\mu pE'(p)+p^2\sigma^2E''(p)/2 \quad p\ge p_{cs} \tag{16}$$

After the enterprise performs bankruptcy liquidation, the equity value is zero, that is: $E(p_{cs})=0$. Therefore, the optimal bankruptcy threshold satisfies the following conditions:

$$E(p_{cs})=0 \tag{17}$$

$$E'(p_{cs})=0 \tag{18}$$

Proposition 3. If financing is through debt, then the equity value before bankruptcy is

$$E(p)=\mathrm{H}(p)-\frac{(1-\tau)c}{r}-\left[\mathrm{H}(p_{cs})-\frac{(1-\tau)c}{r}\right]\left(\frac{p}{p_{cs}}\right)^{\beta} \quad p\ge p_{cs} \tag{19}$$

Where,

$$p_{cs}=(-\mathrm{B}+\sqrt{\mathrm{B}^2-4\mathrm{AC}_c})/(2\mathrm{A})>p_s \tag{20}$$

$\mathrm{C}_c=\beta(Z+c)(r-\mu)(r-2\mu-\sigma^2)$.

Proof. The proof of equation (19) is similar to the **Proposition 1**, so it is omitted. For equation (20), since $\mathrm{C}_c<\mathrm{C}<0\Rightarrow(\mathrm{B}^2-4\mathrm{AC}_c)^{1/2}<(\mathrm{B}^2-4\mathrm{AC})^{1/2}\Rightarrow[-\mathrm{B}+(\mathrm{B}^2-4\mathrm{AC}_c)]^{1/2}/(2\mathrm{A})>[-\mathrm{B}+(\mathrm{B}^2-4\mathrm{AC})^{1/2}]/(2\mathrm{A})$, so $p_{cs}>p_s$.

The expected of the debt value before bankruptcy is

$$D(p)=E_t\int_t^{T_s}ce^{-r(s-t)}ds \quad p\ge p_{cs} \tag{21}$$

Equation (21) can be equivalently converted to the ordinary differential equation:

$$rD(p)=c+\mu pD'(p)+p^2\sigma^2D''(p)/2 \quad p\ge p_{cs} \tag{22}$$

The investment cost comes from debt financing, and bankruptcy is due to insolvency. From the perspective of creditors, the debt value is equal to the after-tax discount value of assets, that is, $D(p_{\mathrm{cs}})$= H(p_{cs}).So, the optimal bankruptcy threshold satisfies the following conditions:

$$D(p_{\mathrm{cs}})=\mathrm{H}(p_{\mathrm{cs}}) \tag{23}$$

$$\lim_{p\to\infty}D(p)=c/r \tag{24}$$

Where, equation (24) represents the debt value is equal to the perpetual value of debt payment, when $p\to\infty$.

Proposition 4. The debt value before bankruptcy is

$$D(p)=\frac{c}{r}-\left[\frac{c}{r}-\mathrm{H}(p_{cs})\right]\left(\frac{p}{p_{cs}}\right)^{\beta} \quad p\ge p_{cs} \tag{25}$$

Proof. Because the general solution of equation (25) is: $D(p)=c/r$-C_$(p/p_{cs})^{\beta}$, according to $D(p_{\mathrm{cs}})$= H(p_{cs}), C_= c/r-H(p_{cs}) can be obtained. Since $\lim_{p\to\infty}D(p)=c/r$=> $\beta=1/\sigma^2\{-(\mu-\sigma^2/2)-[(\mu-\sigma^2/2)^2+2r\sigma^2]^{1/2}\}<0$, so the **Proposition 4** is established.

Because enterprise value is the sum of equity value and debt value, according to equations (19) and (25), the enterprise value is

$$V_c(p)=\mathrm{H}(p)+\frac{\tau c}{r}-\frac{\tau c}{r}\left(\frac{p}{p_{cs}}\right)^{\beta} \qquad p\ge p_{cs} \tag{26}$$

The equity value before executing the growth option satisfies the following model:

$$rE_{c0}(p)=\mu pE_{c0}^{'}(p)+\frac{1}{2}p^2\sigma^2E_{c0}^{''}(p) \qquad p\le p_{ci}$$
$$s.t.\begin{cases}E_{c0}(p_{ci})=V_c(p_{ci})-\mathrm{I}\\ E_{c0}^{'}(p_{ci})=V_c^{'}(p_{ci})\end{cases} \tag{27}$$

Proposition 5. The equity value before executing the growth option is

$$E_{c0}(p)=(V(p_{ci})-\mathrm{I})\left(\frac{p}{p_{ci}}\right)^{\gamma} \qquad p\le p_{ci} \tag{28}$$

Where p_i satisfies the following equation:

$$\left(\mathrm{H}(p_{ci})+\frac{\tau c}{r}-\mathrm{I}\right)\gamma-(1-\tau)\left(\frac{ap_{ci}}{r-\mu}-\frac{2bp_{ci}^2}{r-2\mu-\sigma^2}\right)=(\gamma-\beta)\frac{\tau c}{r}\left(\frac{p_{ci}}{p_{cs}}\right)^{\beta} \tag{29}$$

According to the proof of **Proposition 2**, we can easily get the conclusion of **Proposition 5**.

Sensitivity analysis

Proposition 6. Whether it is equity financing or debt financing, the optimal bankruptcy threshold decreases with the increase of market capacity, and increases with the increase of price-sensitive coefficient.

Proof. Let $M_1=-2r(\beta-2)(r-\mu)$, $M_2=4rZ\beta(\beta-2)(r-\mu)^2(r-2\mu-\sigma^2)$, $M_3=r(\beta-1)(\sigma^2-2\mu-r)$, $M_4=4AC/M_3^2$ and $M_5=-M_3/(2A)$. From equation (10), it can be got that

$$p_s=\frac{B-\sqrt{B^2-M_2b}}{M_1b}\Rightarrow\frac{dp_s}{db}=\frac{2B^2-2\sqrt{B^2-M_2b}-M_2b}{2M_1b^2\sqrt{B^2-M_2b}}=\frac{\left(B-\sqrt{B^2-M_2b}\right)^2}{2M_1b^2\sqrt{B^2-M_2b}}>0,$$

$$p_s=\frac{-M_3a+\sqrt{(M_3a)^2-4AC}}{2A}=\frac{-M_3a+M_3\sqrt{a^2-\frac{4AC}{M_3^2}}}{2A}=M_5\left(a-\sqrt{a^2-M_4}\right)\Rightarrow\frac{dp_s}{da}=M_5\left(1-\frac{a}{\sqrt{a^2-M_4}}\right)<0.$$

Similarly, let $M_6=4r\beta(Z+c)(\beta-2)(r-\mu)^2(r-2\mu-\sigma^2)$ and $M_7=4AC_c/M_3^2$, From equation (20), it can be got that

$$\frac{dp_{cs}}{db}=\frac{\left(B-\sqrt{B^2-M_6b}\right)^2}{2M_1b^2\sqrt{B^2-M_6b}}>0,\ \frac{dp_{cs}}{da}=M_5\left(1-\frac{a}{\sqrt{a^2-M_7}}\right)<0$$

For the analysis of the optimal investment threshold, we will use numerical examples to analyze.

(1) Let[1] r=6%,μ=0,I=50,τ=20%,Z=1,σ=0.2,c=5 and a=10.

According to equations (14) and (29), the relationship between p_i, p_{ci} and b is showed in **Fig.1**.It can be seen from **Fig.1** that the optimal investment threshold increases with the increase of price-sensitive coefficient, under the same price-sensitive coefficient, the optimal investment threshold of equity financing is higher than debt financing.

(2)Let r=6%,μ=0,I=50,τ=20%,Z=1,σ=0.2,c=5 and b=1.

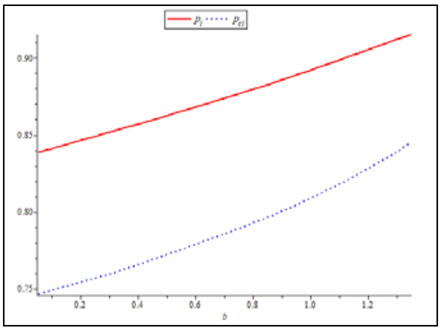

Fig.1 the relationship between p_i,p_{ci} and b

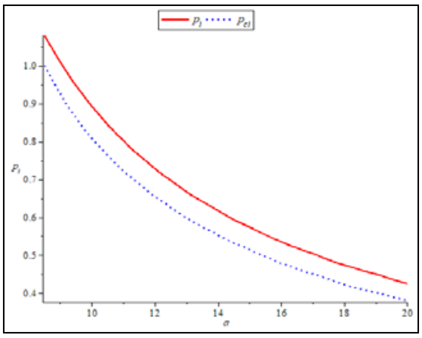

Fig.2 the relationship between p_i,p_{ci} and a

[1] μ=0, Z=1(Bolton, Wang and Yang (2014)); τ=20%, σ=0.2 (Sundaresan, Wang and Yang (2015)).

According to equations (14) and (29), the relationship between p_i, p_{ci} and a is showed in **Fig.2**.It can be seen from **Fig.2** that the optimal investment threshold decreases with the increase of market capacity; under the same market capacity, the optimal investment threshold of equity financing is higher than debt financing.

Conclusion

Our research shows that price-sensitive demand and financing methods have an impact on investment: The optimal bankruptcy threshold of debt financing is higher than equity financing, the optimal bankruptcy threshold and the optimal investment threshold decrease as the market capacity increases, and increase as the price-sensitive coefficient increases. Consequently seen: The bankruptcy risk of debt financing is higher than the equity financing, the bankruptcy risk of the enterprise with higher market capacity is lower, and the enterprise with a higher price-sensitive coefficient has a higher risk of bankruptcy.

Acknowledgements

This paper is supported by the Guangdong Provincial Key Laboratory of Technology and Finance & Big Date Analysis (2017B030301010), Guangzhou philosophy and Social Sciences 13th Five-Year plan 2018(2018GZYB91), Information Platform of Comprehensive Service Centre for Guangdong Technology and Finance (2015B080807015), Guangdong Key Research Base of Technology and Finance (2014B030303005), and Platform of Credit Financing and Trade for Guangdong Technological Enterprises (2014B080807035),Construction on New technology Credit Service Platform Based on O2O Model(2017B080802004).

References

[1]Bolton P, Wang N, Yang J. Investment under uncertainty and the value of real and financial flexibility[R]. National Bureau of Economic Research, 2014.

[2]Brennan M J, Schwartz E S. Optimal financial policy and firm valuation[J]. The journal of finance, 1984, 39(3): 593-607.

[3]Dangl T. Investment and capacity choice under uncertain demand[J]. European Journal of Operational Research, 1999, 117(3): 415-428.

[4]De Giovanni D, Massabò I. Capacity investment under uncertainty: The effect of volume flexibility [J]. International Journal of Production Economics, 2018, 198: 165-176.

[5]Dixit A K, Pindyck R S. The options approach to capital investment[J]. Real Options and Investment under Uncertainty-classical Readings and Recent Contributions. MIT Press, Cambridge, 1995, 6.

[6]Hagspiel V, Huisman K J M, Kort P M. Volume flexibility and capacity investment under demand uncertainty[J]. International Journal of Production Economics, 2016, 178: 95-108.

[7]Kulatilaka N, Perotti E C. Strategic growth options [J]. Management Science, 1998, 44(8): 1021-1031.

[8]McDonald R, Siegel D. The value of waiting to invest[J]. The quarterly journal of economics, 1986, 101(4): 707-727.

[9]Pindyck R S. Irreversibility, uncertainty, and investment[R]. National Bureau of Economic Research, 1990.

[10]Sundaresan S, Wang N, Yang J. Dynamic investment, capital structure, and debt overhang[J]. The Review of Corporate Finance Studies, 2015, 4(1): 1-42.